Multivariable Calculus

$f(x,y) = x \ln(y^2 - x)$ is a function of multiple variables.

$y^2 - x > 0$

$x < y^2$

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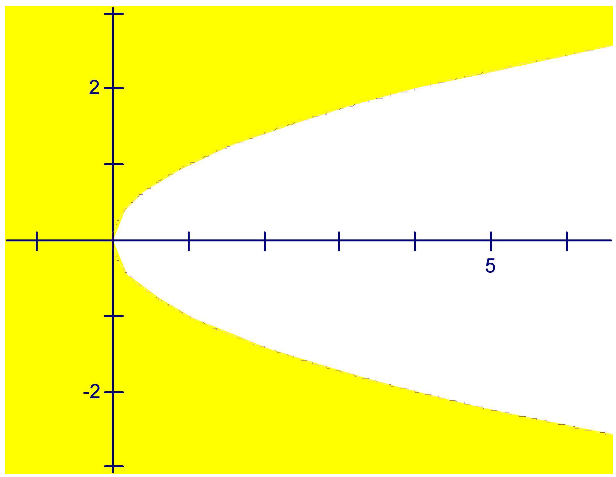

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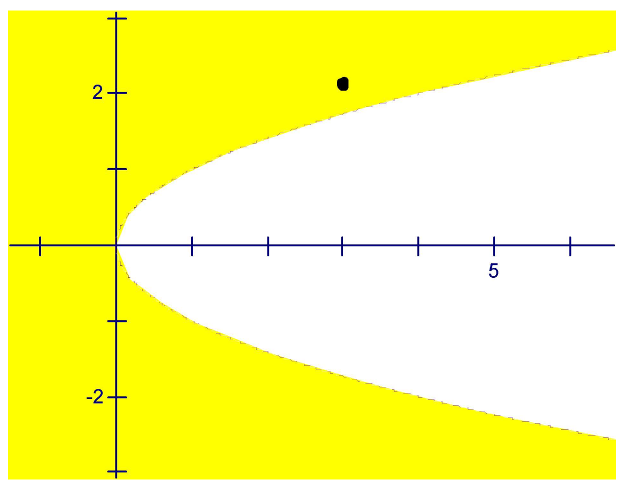


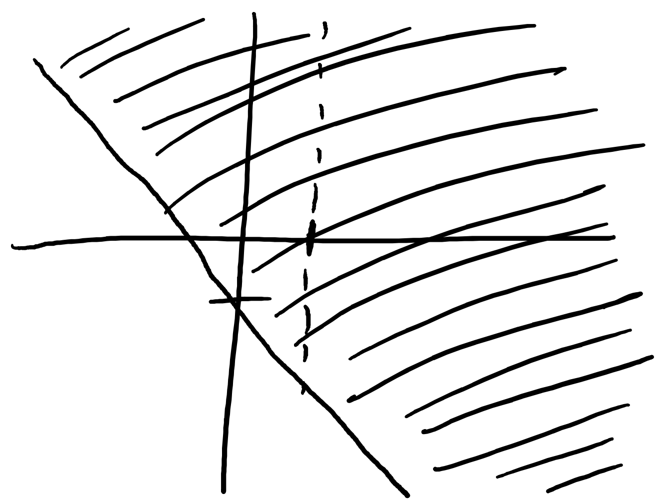
,z

$f(3,2) = 3 \ln (2^2 - 3) = 3 \ln (1) = 0$

<u>Ex.</u> Find the domain of $f(x,y)=\dfrac{\sqrt{x+y+1}}{x-1}$

$x+y+1 \geq 0$ $\quad\quad\quad\quad$ $x-1 \neq 0$

$y \geq -x-1$ $\quad\quad\quad\quad$ $x \neq 1$

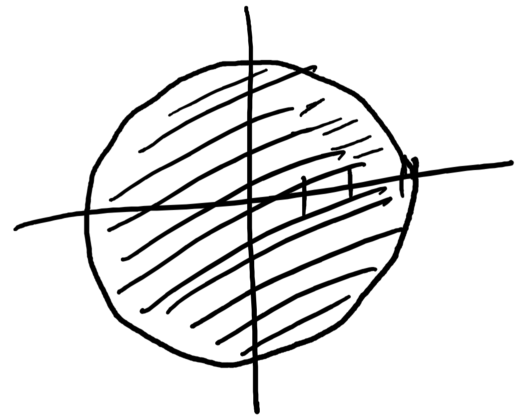


Ex. Find the domain and range of

$$g(x,y)=\sqrt{9-x^2-y^2} = \sqrt{9-(x^2+y^2)}$$

$$9-x^2-y^2 \geq 0$$

$$x^2+y^2 \leq 9$$



$$0 \leq g(x,y) \leq 3$$

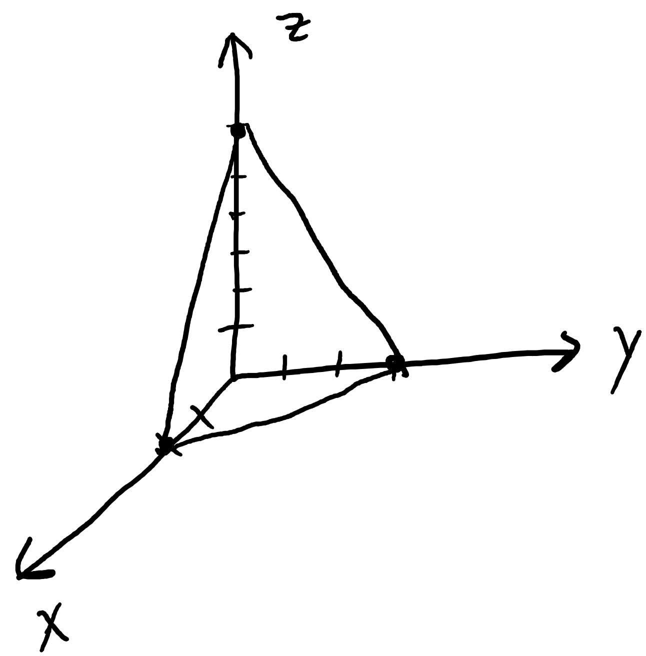
<u>Ex.</u> Sketch the graph of $f(x,y) = 6 - 3x - 2y$.

$z = 6 - 3x - 2y$

$3x + 2y + z = 6$

domain: $\mathbb{R}^2$

range: $\mathbb{R}$



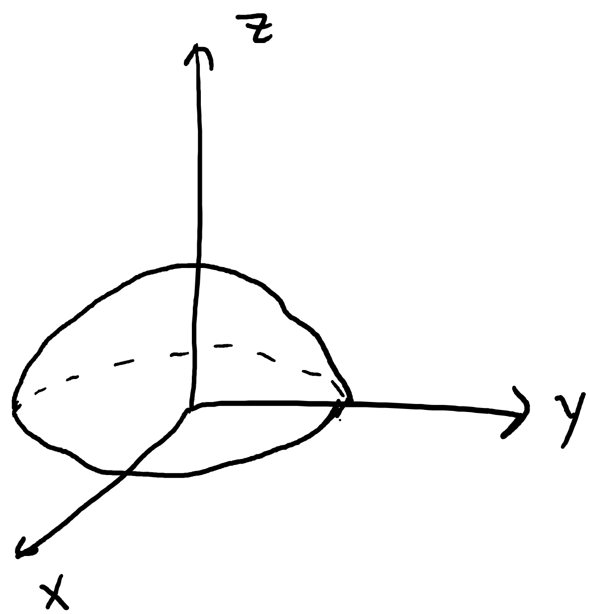
This is a linear function of two variables.

Ex. Sketch the graph of $g(x,y)=\sqrt{9-x^2-y^2}$

$z=\sqrt{9-x^2-y^2}$

$z^2=9-x^2-y^2$

$x^2+y^2+z^2=9$

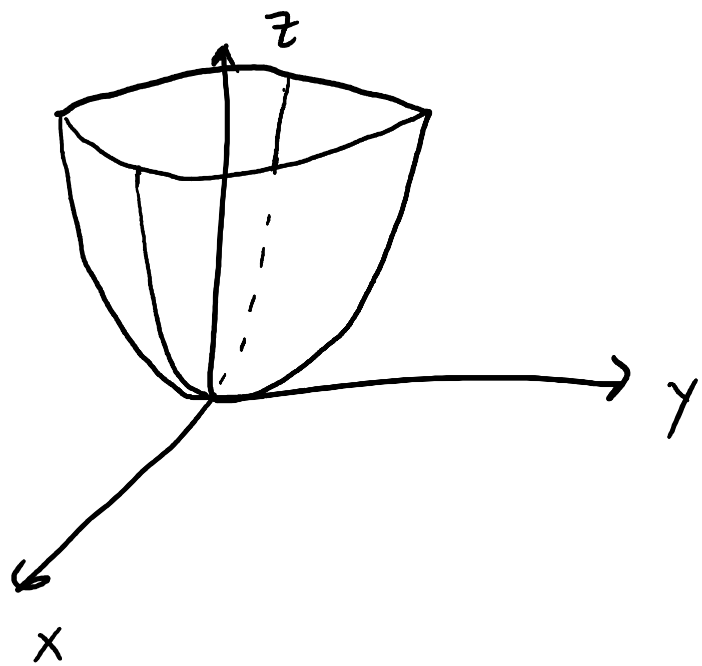


Ex. Find the domain and range of $f(x,y) = 4x^2 + y^2$ and identify the graph.

$z = 4x^2 + y^2$

domain: $\mathbb{R}^2$

range: $z \geq 0$

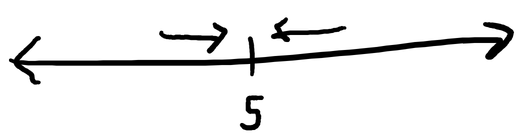


Ex. $\lim_{(x,y)\to(1,2)}\left(x^2y^3 - x^3y^2 + 3x + 2y\right)$

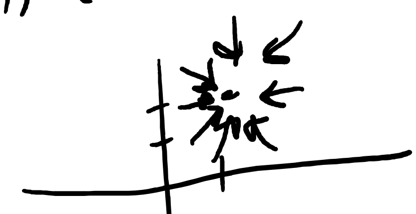
$= 1^2 2^3 - 1^3 2^2 + 3\cdot 1 + 2\cdot 2$

$= 8 - 4 + 3 + 4 = \boxed{11}$

$\lim_{x\to 5}$



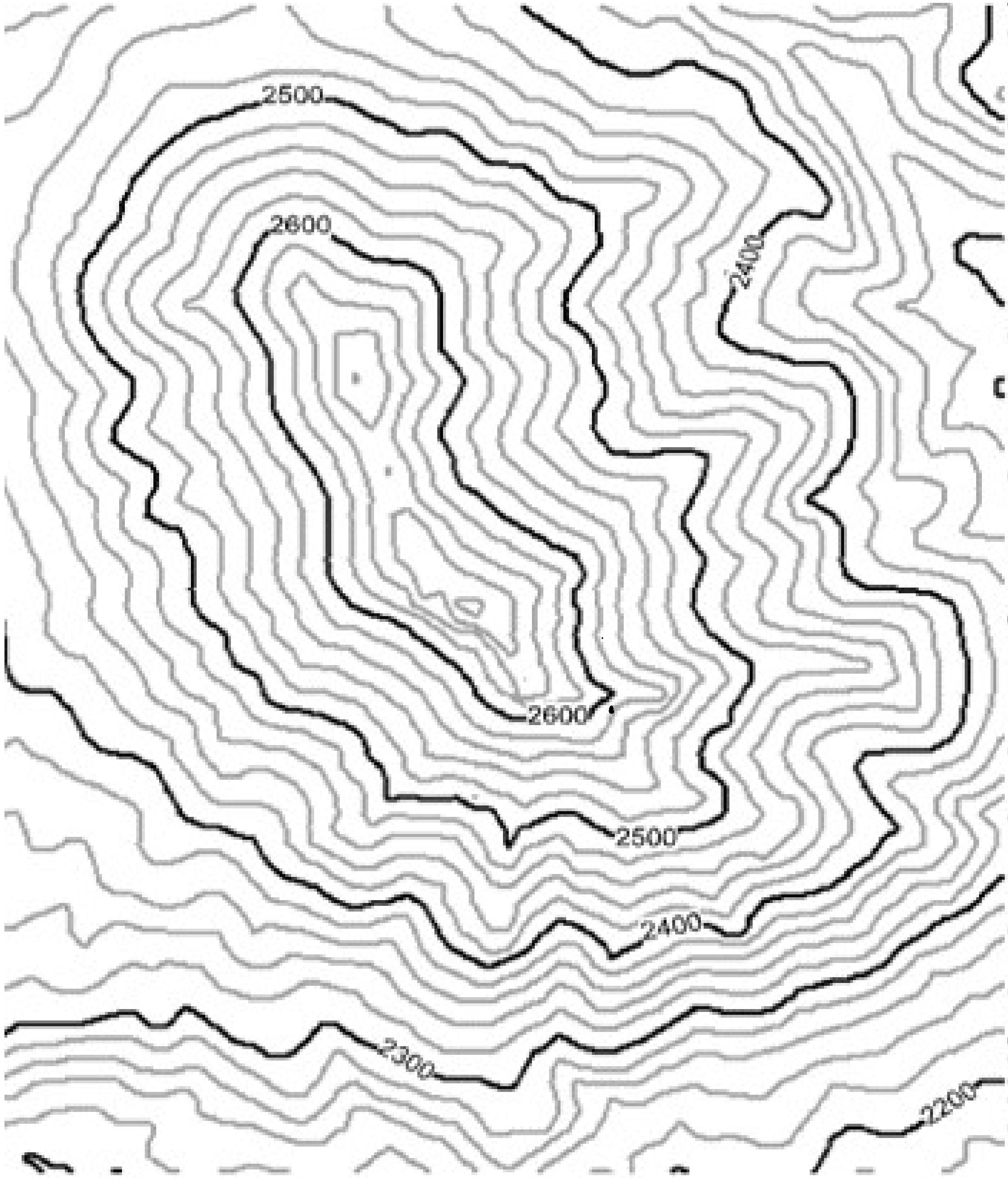
$\lim_{(x,y)\to(1,2)}$



This is a polynomial function of two variables.

When trying to sketch multivariable functions, it can convenient to consider level curves (contour lines). These are 2-D representations of all points where f has a certain value.

→This is what you do when drawing a topographical map.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<u>Ex.</u> Sketch the level curves of $f(x,y)=\sqrt{9-x^2-y^2}$ for $k = 0, 1, 2,$ and 3.

$k=0: \quad 0=\sqrt{9-x^2-y^2}$

$x^2+y^2=9$

$k=1: \quad 1=\sqrt{9-x^2-y^2}$

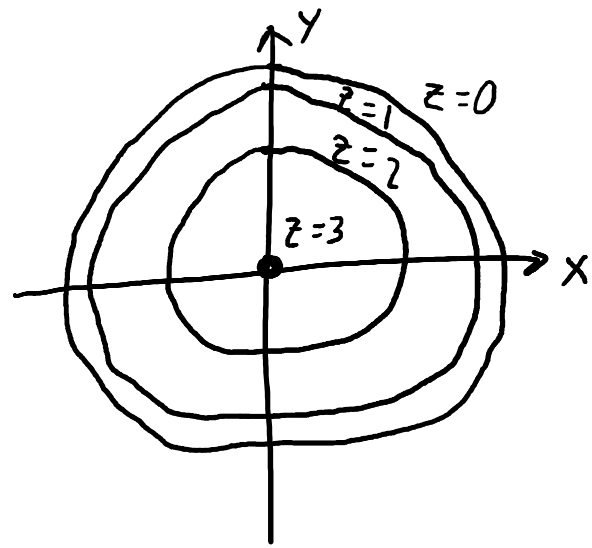
$x^2+y^2=8$

$k=2: \quad 2=\sqrt{9-x^2-y^2}$

$x^2+y^2=5$

$k=3: \quad 3=\sqrt{9-x^2-y^2}$

$x^2+y^2=0$



Ex. Sketch some level curves of $f(x,y) = 4x^2 + y^2$

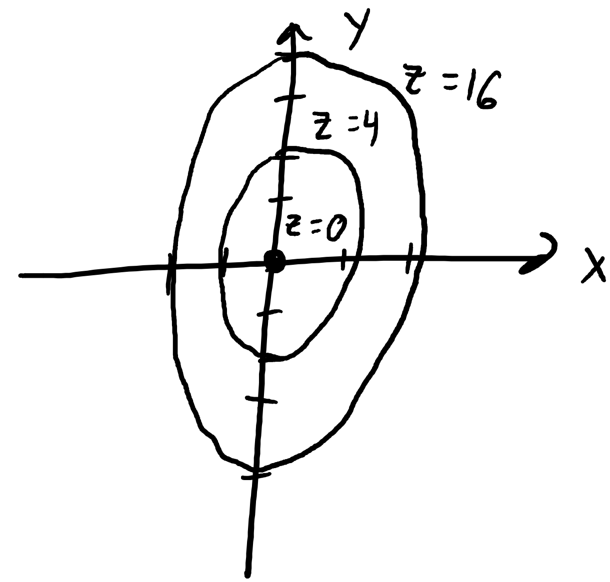
$k=0: \quad 0 = 4x^2 + y^2$

$k=4: \quad 4 = 4x^2 + y^2$

$1 = \frac{x^2}{1} + \frac{y^2}{4}$

$k=16: \quad 16 = 4x^2 + y^2$

$1 = \frac{x^2}{4} + \frac{y^2}{16}$



A function like $T(x,y,z)$ could represent the temperature at any point in the room.

Ex. Find the domain of $f(x,y,z) = \ln(z - y)$.

$z - y > 0$

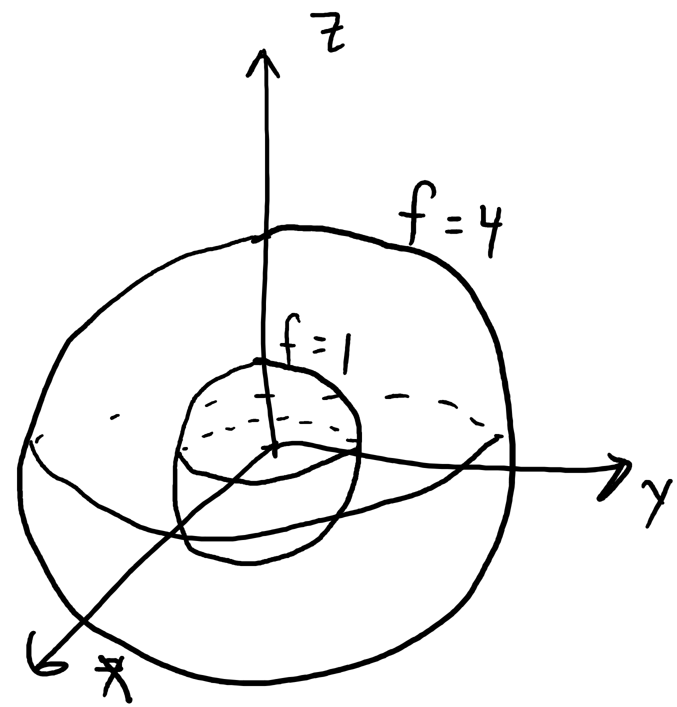
$z > y$

Ex. Identify the level curves of

$$f(x,y,z) = x^2 + y^2 + z^2$$

$k=1$: $\quad 1 = x^2 + y^2 + z^2$

$k=4$: $\quad 4 = x^2 + y^2 + z^2$



Partial Derivatives

A <u>partial derivative</u> of a function with multiple variables is the derivative with respect to one variable, treating other variables as constants.

If $z = f(x,y)$, then

wrt x: $$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = z_x = \frac{\partial z}{\partial x}$$

wrt y: $$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = z_y = \frac{\partial z}{\partial y}$$

<u>Ex.</u> Let $f(x,y) = xe^{x^2 y}$, find f_x and f_y and evaluate them at (1,ln 2).

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$f_x = x \cdot e^{x^2 y} \cdot 2xy + e^{x^2 y} \cdot 1$

$f_x(1, \ln 2) = 1\, e^{\ln 2} \cdot 2 \cdot 1 \ln 2 + e^{\ln 2}$

$= 4 \ln 2 + 2$

$f_y = x e^{x^2 y} \cdot x^2$

$f_y(1, \ln 2) = 1\, e^{\ln 2} \cdot 1 = 2$

z_x and z_y are the slopes in the x- and y-direction

Ex. Find the slopes in the x- and y-direction of the surface $f(x,y) = \frac{-1}{2}x^2 - y^2 + \frac{25}{8}$ at $\left(\frac{1}{2}, 1, 2\right)$

$f_x = -x \qquad f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2}$

$f_y = -2y \qquad f_y\left(\frac{1}{2}, 1\right) = -2$

Ex. For $f(x,y) = x^2 - xy + y^2 - 5x + y$, find all values of x and y such that f_x and f_y are zero simultaneously.

$f_x = 2x - y - 5 = 0$ $\qquad$ $f_y = -x + 2y + 1 = 0$

$2x - y = 5 \longrightarrow 2x - y = 5$

$-x + 2y = -1 \xrightarrow{\times 2} -2x + 4y = -2$

$3y = 3$

$y = 1$

$2x - 1 = 5$

$2x = 6$

$x = 3$

Ex. Let $f(x,y,z) = xy + yz^2 + xz$, find all partial derivatives.

$$f_x = y + z$$

$$f_y = x + z^2$$

$$f_z = 2yz + x$$

Higher-order Derivatives

$$\frac{\partial}{\partial x} f_x = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\left.\begin{aligned} \frac{\partial}{\partial y} f_x &= \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \\ \frac{\partial}{\partial x} f_y &= \frac{\partial^2 f}{\partial x \partial y} = f_{yx} \end{aligned}\right\} \text{mixed partial derivatives}$$

$$\frac{\partial}{\partial y} f_y = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

<u>Ex.</u> Find the second partial derivatives of $f(x,y) = 3xy^2 - 2y + 5x^2y^2$.

$f_x = 3y^2 + 10xy^2$ $\qquad$ $f_y = 6xy - 2 + 10x^2y$

$f_{xx} = 10y^2$ $\qquad$ $f_{xy} = 6y + 20xy$ $\qquad$ $f_{yx} = 6y + 20xy$ $\qquad$ $f_{yy} = 6x + 10x^2$

$$f_{xy} = f_{yx}$$

<u>Ex.</u> Let $f(x,y) = ye^x + x \ln y$, find f_{xyy}, f_{xxy}, and f_{xyx}.

$$f_x = ye^x + \ln y$$

$$f_{xy} = e^x + \frac{1}{y}$$

$$f_{xyy} = \frac{-1}{y^2} \qquad f_{xyx} = e^x = f_{xxy}$$

A <u>partial differential equation</u> can be used to express certain physical laws.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow u_{xx} + u_{yy} = 0$$

This is <u>Laplace's equation</u>. The solutions, called <u>harmonic equations</u>, play a role in problems of heat conduction, fluid flow, and electrical potential.

Ex. Show that $u(x,y) = e^x \sin y$ is a solution to Laplace's equation.

$u_x = e^x \sin y$

$u_{xx} = e^x \sin y$

$u_y = e^x \cos y$

$u_{yy} = -e^x \sin y$

$u_{xx} + u_{yy} = 0$ ✓

$3x - 5 = 7$

$x = 4$

Another PDE is called the <u>wave equation</u>:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad u(t,x)$$

Solutions can be used to describe the motion of waves such as tidal, sound, light, or vibration.

The function $u(x,t) = \sin(x - at)$ is a solution.